An analysis of input design for state-space models

Patricio E. Valenzuela[(1)], Johan Dahlin[(2)], Cristian R. Rojas[(1)], Thomas B. Schön[(3)], Håkan Hjalmarsson[(1)]

Problem Formulation

The system Consider

$$x_{t+1} = Ax_t + Bu_t + e_t \quad (\text{or } x_t \sim f_\theta(x_t|x_{t-1}, u_{t-1}))$$
$$y_t = Cx_t + Du_t + n_t \quad (\text{or } y_t \sim h_\theta(y_t|x_t, u_t))$$

where $x_t, e_t \in \mathbb{R}^{n_x}$, $y_t, u_t, n_t \in \mathbb{R}$, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times 1}$, $C \in \mathbb{R}^{1 \times n_x}$, $D \in \mathbb{R}$. $\{e_t\}$ and $\{n_t\}$ are Gaussian white noise sequences with zero mean and variance Σ_e and Σ_n, respectively.

Objective Design $\mathcal{U}_{n_{\text{seq}}} := (u_{n_{\text{seq}}}, \ldots, u_1)$ as a realization of a stationary process of memory n_m maximizing

$$\mathcal{I}_F := \mathbf{E}\{(\nabla_\theta \log p_\theta(y_{1:n_{\text{seq}}}))^2 | u_{1:n_{\text{seq}}}\}$$

Estimating the Information matrix

The Information matrix is defined as the variance of the score function and can be estimated by the Monte Carlo covariance of the score taken over realisations of the system given an input

$$\widehat{\mathcal{I}}_F^{(i)} = \mathbf{V}\left\{\widehat{\mathcal{S}}(\theta) | \{u_t^{(i)}\}\right\},$$

where $\widehat{\mathcal{S}}(\theta)$ denotes the score estimated using Fisher's identity with the form

$$\widehat{\mathcal{S}}(\theta) = \int \nabla_\theta \log p_\theta(x_{1:T}, y_{1:T}) \widehat{p}_\theta(x_{1:T}|y_{1:T}) dx_{1:T},$$

where $\widehat{p}_\theta(x_{1:T}|y_{1:T})$ denotes the joint empirical filtering distribution obtained from a particle smoother.

The input design method

1. Compute all the elementary cycles of $\mathcal{G}_{\mathcal{C}^{(n_m-1)}}$.
2. Compute all the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$ from the elementary cycles of $\mathcal{G}_{\mathcal{C}^{(n_m-1)}}$.
3. Generate the input signals $\{u_t^{(i)}\}_{t=0}^{t=N}$ from the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$, for each $i \in \{1, \ldots, n_\mathcal{V}\}$.
4. For each $i \in \{1, \ldots, n_\mathcal{V}\}$, approximate $\mathcal{I}_F^{(i)}$ by using particle methods.
5. Define $\gamma := \{\alpha_1, \ldots, \alpha_{n_\mathcal{V}}\} \in \mathbb{R}^{n_\mathcal{V}}$.

   Find $\gamma^{\text{opt}} := \{\alpha_1^{\text{opt}}, \ldots, \alpha_{n_\mathcal{V}}^{\text{opt}}\}$ by solving

   $$\gamma^{\text{opt}} := \arg \max_{\gamma \in \mathbb{R}^{n_\mathcal{V}}} h(\widehat{\mathcal{I}}_F(\gamma))$$

   where

   $$\widehat{\mathcal{I}}_F(\gamma) := \sum_{i=1}^{n_\mathcal{V}} \alpha_i \widehat{\mathcal{I}}_F^{(i)}$$
   $$\sum_{i=1}^{n_\mathcal{V}} \alpha_i = 1$$
   $$\alpha_i \geq 0, \text{ for all } i \in \{1, \ldots, n_\mathcal{V}\}$$

6. The optimal pmf p^{opt} is given by

   $$p^{\text{opt}} = \sum_{i=1}^{n_\mathcal{V}} \alpha_i^{\text{opt}} v_i$$

Generation of an input sequence

1. Associate each entry in $\Pi^s \in \mathbb{R}^{(c_{\text{seq}})^{n_m}}$ to one possible value of p^{opt}.
2. Build a transition probability matrix $P \in \mathbb{R}^{(c_{\text{seq}})^{n_m} \times (c_{\text{seq}})^{n_m}}$ such that

   $$\Pi^s = P\Pi^s$$

3. Run

   $$\Pi_{k+1} = P\Pi_k$$

   with random initial state Π_0.

Example Consider

$$x_{t+1} = a_0 x_t + u_t + e_t$$
$$y_t = c_0 x_t + n_t$$

with $a_0 = 0.5$, $c_0 = 1$, $\Sigma_e = 1$, and $\Sigma_n = 0.1$. We design experiments to identify a_0, and Σ_e, with $n_{\text{seq}} = 5 \cdot 10^5$, $n_m = 1$, and $\mathcal{C} = \{-1, 0, 1\}$. The optimal experiments maximize $h(\widehat{\mathcal{I}}_F(\gamma)) = \det(\widehat{\mathcal{I}}_F(\gamma))$, and $h(\widehat{\mathcal{I}}_F(\gamma)) = -\text{tr}\left\{(\widehat{\mathcal{I}}_F(\gamma))^{-1}\right\}$.

Cost / Input	Uniform	Binary	Optimal
$\det\left\{\widehat{\mathcal{I}}_F\right\}$	7733.81	12779.62	12457.66
$\text{tr}\left\{(\widehat{\mathcal{I}}_F)^{-1}\right\}$	0.0270	0.0245	0.0247

References

[1] P.E. Valenzuela, C.R. Rojas and H. Hjalmarsson. "Optimal input design for non-linear dynamic systems: a graph theory approach". to be presented at *52nd Conference on Decision and Control*, Florence, Italy, 2013.

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(1) Department of Automatic Control, School of Electrical Engineering, KTH, Stockholm, Sweden
(2) Division of Automatic Control, Linköping University, Linköping, Sweden
(3) Division of Systems and Control, Uppsala University, Uppsala, Sweden